Concluding Remarks

A global-local approximation based on a linear approximation to a scaling factor has been presented. The approximation permits us to use a global approximation based on a simple model of our problem to extend the range of usefulness of derivative-based approximations to a more refined model. The method was demonstrated for a simple beam example with a crude and more refined finite element model.

Acknowledgment

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Algorithm for Modification of Parameters in Vibrating Systems

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Nomenclature

$\mathfrak{D}_B A$ = derivative of a $p \times q$ matrix function A with respect to an $s \times t$ matrix argument B (see Ref. 2)

$$\frac{\partial A}{\partial B} = \left[\frac{\partial A}{\partial b_{kl}}\right] \quad (k = 1,\ldots,s;\ l = 1,\ldots,t)$$

where

$$\frac{\partial A}{\partial b_{kl}} = \left[\frac{\partial a_{ij}}{\partial b_{kl}}\right] \quad (i = 1,\ldots,p;\ j = 1,\ldots,q)$$

$A \otimes B$ = Kronecker product of a $p \times q$ matrix A and an $s \times t$ matrix B (see Ref. 3)

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1q}B \\ a_{21}B & a_{22}B & \ldots & a_{2q}B \\ \vdots & \vdots & \ldots & \vdots \\ a_{p1}B & a_{p2}B & \ldots & a_{pq}B \end{bmatrix}$$

I = unit matrix

E^{ij} = ijth elementary matrix; all elements zero except for 1 at the ijth position

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Introduction

THE importance of obtaining sensitivities for an eigenvalue problem stems from the fact that partial derivatives with respect to system parameters are extremely important for effecting efficient design modifications for given situations, for gaining insight into the reasons for discrepancies between structural analyses and dynamic tests, and for indicating system model changes that will improve correlations between analyses and tests. Knowledge of sensitivities can be very useful in the modification of system parameters; however, the changes of these parameters that realize a system possessing prescribed eigenvalues and eigenvectors are still unknown.

This Note presents and demonstrates an efficient method of determining changes in system parameters for predetermined changes of eigenvalues and eigenvectors.

Eigenvalue Problem of Undamped Systems

The eigenvalue problem of a vibrating system described by the equation

$$M\ddot{y} + Ky = 0 \tag{1}$$

where the matrices M and K are symmetric and positive definite and can be written in the matrix form

$$(K - \omega^2 M)\, y = 0 \tag{2}$$

Let us suppose that the system (1) has n distinct eigenvalues. Then we have n linearly independent eigenvectors y_i corresponding to the eigenvalues λ_i. The eigenvalues form the spectral matrix of order n, Λ, and the eigenvectors can be arranged in an $n \times n$ square matrix Y, called the modal matrix. These matrices satisfy the orthogonality given by

$$Y^T M\, Y = I \tag{3}$$

$$Y^T K\, Y = \Lambda \tag{4}$$

Modifications of System Parameters

The modification problem consists of determining changes to the parameters of system (1) represented by the matrices M and K, which realize a new system possessing predetermined eigenvalues and eigenvectors represented by Λ and Y. In other words, we want to determine ΔM and ΔK so that the new system $M^* = M + \Delta M$, $K^* = K + \Delta K$ has prescribed eigenvalues $\Lambda^* = \Lambda + \Delta\Lambda$ and eigenvectors $Y^* = Y + \Delta Y$.

It is seen from Eqs. (3) and (4) that the matrices M and K are functions of the matrices Λ and Y, e.g.,

$$M = M(\Lambda, Y) \tag{5}$$

$$K = K(\Lambda, Y) \tag{6}$$

We will assume that the elements of the matrices Λ and Y change and that these changes are small. We want to determine increments ΔM and ΔK of parameters from a new set of the eigenvalues Λ^* and the eigenvectors Y^*. We will use for that purpose the Taylor expansion.[1] According to this expansion, for sufficiently small increments of parameters, only the first term in the expansions need by retained, i.e.,

$$M^* = M + \mathfrak{D}_{\lambda T} M \cdot (\Delta\lambda \otimes I) + \mathfrak{D}_{yT} K \cdot (\Delta y \otimes I) \tag{7}$$

$$K^* = K + \mathfrak{D}_{\lambda T} M \cdot (\Delta\lambda \otimes I) + \mathfrak{D}_{yT} K \cdot (\Delta y \otimes I) \tag{8}$$

where the vector $\lambda^T = \lfloor \lambda_1, \lambda_2, \ldots, \lambda_n \rfloor$, and y^T is the column transformation of Y. The matrices $\mathfrak{D}_{\lambda^T} M$, $\mathfrak{D}_{\lambda^T} K$, etc., are unknown. We will determine these matrices.

From the definition

$$\mathfrak{D}_{\lambda^T} M = [\mathfrak{D}_{\lambda_1} M, \ldots, \mathfrak{D}_{\lambda_n} M] \tag{9}$$

$$\mathfrak{D}_{y^T} M = [\mathfrak{D}_{y_{11}} M, \ldots, \mathfrak{D}_{y_{nn}} M] \cdots \text{etc} \tag{10}$$

We have, from Eqs. (3) and (4),

$$M = (Y^T)^{-1} Y^{-1} \tag{11}$$

$$K = (Y^T)^{-1} \Lambda Y^{-1} \tag{12}$$

The differentiating of Eqs. (11) and (12) with respect to λ_i gives

$$\mathfrak{D}_{\lambda_i} M = 0 \tag{13}$$

$$\mathfrak{D}_{\lambda_i} K = (Y^T)^{-1} E^{ii} Y^{-1} \tag{14}$$

and, hence,

$$\mathfrak{D}_{\lambda^T} M = 0 \tag{15}$$

$$\mathfrak{D}_{\lambda^T} K = [(Y^T)^{-1} E^{11} Y^{-1}, \ldots, (Y^T)^{-1} E^{nn} Y^{-1}] \tag{16}$$

The differentiating of Eq. (3) with respect to y_{ij} gives

$$Y^T \mathfrak{D}_{y_{ij}} M \cdot Y = -(E^{ji} M Y + Y^T M E^{ij}) \tag{17}$$

and, finally,

$$\mathfrak{D}_{y_{ij}} M = -(Y^T)^{-1} (E^{ji} M Y + Y^T M E^{ij}) Y^{-1} \tag{18}$$

Proceeding analogously, one obtains

$$\mathfrak{D}_{y_{ij}} K = -(Y^T)^{-1} (E^{ji} K Y + Y^T K E^{ij}) Y^{-1} \tag{19}$$

These matrices will be called inverse sensitivity matrices. They show how the matrices M and K change when the matrices Λ and Y change.

Examples

Numerical examples illustrate the theory and its possibilities. Consider the two-degree-of-freedom M,K system where the matrices M and K are defined as follows:

$$M = \begin{bmatrix} 5 & 1.5 \\ 1.5 & 1 \end{bmatrix} \qquad K = 10^3 \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

The eigenvalues and normalized eigenvectors of the system are, respectively,

$$\lambda_1 = 106.9, \qquad \lambda_2 = 6802.2$$

$$y_1^N = \begin{bmatrix} 0.3207 \\ 0.3659 \end{bmatrix}, \qquad y_2^N \begin{bmatrix} -0.5107 \\ 1.2978 \end{bmatrix}$$

The vectors λ^T and y^T have the form, respectively,

$$\lambda^T = \lfloor \lambda_1, \lambda_2 \rfloor, \qquad y^T = \lfloor y_{11}, y_{21}, y_{12}, y_{22} \rfloor$$

To explain the method presented here we modify expressions (7) and (8). One obtains, for the two-degree-of-freedom system

$$\mathfrak{D}_{\lambda^T} M (\Delta\lambda \otimes I) = \mathfrak{D}_{\lambda_1} M \cdot \Delta\lambda_1 + \mathfrak{D}_{\lambda_2} M \cdot \Delta\lambda_2 \tag{20}$$

$$\mathfrak{D}_{y^T} M (\Delta y \otimes I) = \mathfrak{D}_{y_{11}} M \cdot \Delta y_{11} + \cdots + \mathfrak{D}_{y_{22}} M \cdot \Delta y_{22} \tag{21}$$

Let the new set of spectral data be as follows (to simplify computations we assume $\Delta\lambda_1 = \Delta y_{12} = \Delta y_{22} = 0$):

$$\lambda_1^* = \lambda_1, \qquad \lambda_2^* = 7302$$

$$y_1 = \begin{bmatrix} 0.3707 \\ 0.4259 \end{bmatrix}, \qquad y_2^* = y_2$$

We want to find ΔM and ΔK so that the modified system has these new spectral data. One obtains from Eq. (7), taking into account Eq. (20),

$$\Delta M = M_{\lambda_1} + M_{\lambda_2} + M_{y_{11}} + \cdots + M_{y_{22}}$$

where M_{λ_i} is the increment of the matrix M after changes of the eigenvalues λ_i, and $M_{y_{ij}}$ is the increment of this matrix after the changes of components of the normalized eigenvectors.

We have the considered system

$$\Delta M = \underbrace{\begin{bmatrix} -1.0761 & -0.3731 \\ -0.3731 & -0.1270 \end{bmatrix}}_{M_{y_{11}}} + \underbrace{\begin{bmatrix} -0.3874 & -0.2054 \\ -0.2054 & -0.1016 \end{bmatrix}}_{M_{y_{21}}}$$

$$= \begin{bmatrix} -1.4635 & -0.5785 \\ -0.5785 & -0.2286 \end{bmatrix}$$

$$\Delta K = \underbrace{\begin{bmatrix} 184.1 & -161.3 \\ -161.3 & -141.4 \end{bmatrix}}_{K_{\lambda_2}} + \underbrace{\begin{bmatrix} -645.7 & -88.2 \\ 88.2 & -169.4 \end{bmatrix}}_{K_{y_{11}}}$$

$$+ \underbrace{\begin{bmatrix} 516.5 & -156.6 \\ -156.6 & -203.3 \end{bmatrix}}_{K_{y_{21}}} = \begin{bmatrix} 55.0 & -229.8 \\ -229.8 & 107.5 \end{bmatrix}$$

It is easy to verify that the new eigenvalue problem is satisfied.

Let us consider now a five-degree-of-freedom system described by matrices

$$M = \begin{bmatrix} 5 & 1 & 3 & 0 & 2 \\ 1 & 3 & 2 & 6 & 5 \\ 3 & 2 & 3 & 4 & 0 \\ 0 & 6 & 4 & 1 & 8 \\ 2 & 5 & 0 & 8 & 2 \end{bmatrix}$$

$$K = 10^3 \begin{bmatrix} 1 & -2 & 2 & -1 & 3 \\ -2 & 4 & -2 & 5 & -1 \\ 2 & -2 & 1 & -3 & 2 \\ -1 & 5 & -3 & 4 & -5 \\ 3 & -1 & 2 & -5 & 3 \end{bmatrix}$$

The eigenvalues and eigenvectors of the system are as follows

$$\Lambda = \text{diag}\ (1.15, 3.24, 8.01, 11.89, 19.14)$$

$$Y = \begin{bmatrix} 0.38 & -0.56 & 0.50 & 0.28 & 0.37 \\ -0.15 & 0.37 & 0.76 & 0.36 & 0.42 \\ 0.87 & 0.39 & -0.29 & 0.71 & -0.15 \\ 0.25 & -0.12 & 0.45 & -0.18 & 1.21 \\ 0.49 & 1.07 & -0.98 & 0.57 & -0.66 \end{bmatrix}$$

We will reconstruct the system possessing a set of spectral data of the form

$$\Lambda^* = \text{diag}\ (1.0,\ 3.5,\ 8.2,\ 11.5,\ 20.0)$$

and

$$Y^* = \begin{bmatrix} 0.40 & -0.56 & 0.50 & 0.30 & 0.37 \\ -0.12 & 0.37 & 0.76 & 0.40 & 0.40 \\ 0.85 & 0.42 & -0.30 & 0.70 & -0.15 \\ 0.25 & -0.10 & 0.45 & -0.20 & 1.21 \\ 0.50 & 1.00 & -0.98 & 0.57 & -0.66 \end{bmatrix}$$

Proceeding thus, one finds

$$M = \begin{bmatrix} 4.78 & 1.15 & 3.21 & 0.22 & 1.72 \\ 1.15 & 3.24 & 2.17 & 5.81 & 5.19 \\ 3.21 & 2.17 & 2.92 & 4.23 & 0.17 \\ 0.22 & 5.81 & 4.23 & 0.83 & 8.31 \\ 1.72 & 5.19 & 0.17 & 8.31 & 1.86 \end{bmatrix}$$

$$K = 10^3 \begin{bmatrix} 1.14 & -1.78 & 1.91 & -0.89 & 3.28 \\ -1.78 & 4.31 & -2.19 & 5.42 & -1.21 \\ 1.91 & -2.49 & 0.87 & -2.71 & 2.09 \\ -0.89 & 5.42 & -2.71 & 3.62 & -4.77 \\ 3.28 & -1.21 & 2.09 & -4.77 & 2.72 \end{bmatrix}$$

One should emphasize that, generally, the effectiveness of the procedure presented here does not depend on the order of the system. Of course, the number of mathematical operations increases when the order of the system increases, but the algorithm does not depend on the number of degrees of freedom.

Conclusions

A method has been proposed for determining modified parameters in symmetric undamped vibratory systems. The method can be applied to a large class of systems with distinct eigenvalues.

The algorithm demonstrated in this Note shows how to change system parameters in order to obtain predetermined spectral data. It is especially useful in practice when we must repeat many times the computations for the same system, but for a different set of eigenvalues and eigenvectors. For the given modified system, after computations of the inverse sensitivities, the increments of the parameters ΔM and ΔK are determined very fast by way of a simple number of matrix multiplications [see Eqs. (20) and (21)]. The method is useful in modifications of complex structures with a large number of degrees of freedom because it is much faster than classical calculations of parameter changes computed on the basis of Eqs. (11) and (12).

One should emphasize that the algorithm presented here allows comparison of the increments ΔM and ΔK produced by the changes of $\Delta\Lambda$ and ΔY, taking into account shares of the individual increments $\Delta\lambda_i$ and Δy_{ij}.

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Error Estimators for Eigenvalues Computed from Discretized Models of Vibrating Structures

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Introduction

NUMERICAL analysis of a solid, plate, or shell structure requires discretization, e.g., by finite elements. Realizing that discretization is accompanied by modeling error, the analyst seeks a level of mesh refinement that is adequate but not excessive. An error estimator for time-independent stress analysis, based on the discrepancy between smoothed and unsmoothed stress fields, has been suggested.[1] The present Note suggests two analogous estimators for error in computed natural frequencies. Both are postprocessing schemes that use the Rayleigh quotient.

In published work related to the present study, the analytical tools are usually called perturbation theory or design sensitivity analysis (e.g., Refs. 2 and 3). There the goal is to predict the change in eigenvalue associated with an actual change in structure stiffness, mass, or support conditions. In the present Note, we make use of the change in eigenvalue associated with changing the representation of stiffness and/or mass, without any changes in the actual structure or in the number of elements used. The motivation is that two different but legitimate representations of stiffness and/or mass will yield two different frequencies for a given mode; hence, the discrepancy between the two frequencies can serve as a measure of computational error for that mode. The second eigenvalue could be obtained by complete resolution of the eigenproblem, but computational economy is greatly enhanced by obtaining it from the Rayleigh quotient instead.

Method 1

For undamped free vibration, the eigenvalue problem is

$$([K_o] - \omega_o^2\,[M_o])\ \{D_o\} = \{0\} \tag{1}$$

where ω_o represents natural frequency, $\{D_o\}$ is a vector of amplitudes of degrees of freedom, and $[K_o]$ and $[M_o]$ are, respectively, the structure stiffness and mass matrices. Subscript o indicates "original," i.e., prior to any modifications of the finite element modeling. Equation (1) is now solved by any convenient eigensolver to obtain natural frequency ω_{oi} and the associated eigenvector $\{D_o\}_i$ for each mode i of interest. Thus far the calculations are standard.

If $[K_o]$, $[M_o]$, and $\{D_o\}_i$ were now entered into the Rayleigh quotient, we would, of course, obtain the known result ω_{oi}^2. Instead, without changing the mesh layout or the number of degrees of freedom, let us evaluate the Rayleigh quotient using an alternative mass matrix $[M_R]$ in place of $[M_o]$, so as to obtain a second natural frequency ω_{Ri}.

$$\omega_{Ri}^2 = \frac{\{D_o\}_i^T\,[K_o]\,\{D_o\}_i}{\{D_o\}_i^T\,[M_R]\,\{D_o\}_i} \tag{2}$$

For example, $[M_o]$ might be the lumped formulation (i.e., diagonal, as from particle masses) and $[M_R]$ the consistent formulation (i.e., derived using the same shape functions used to derive the stiffness matrix). It is well known that frequencies usually converge from below when $[M]$ is lumped and

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